Positive Solution of a Nonlinear Quadratic Integral Equations

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Abstract

In the present paper we prove the existence, approximations and positive solutions for a nonlinear quadratic integral equation. By using Dhage iteration method in partially ordered normed linear spaces.

Subject Classification

Quadratic integral equation; approximate positive solution; fixed point theorem.

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Introduction

The quadratic integral equations have been studied by some autors see See Ar- gyros [1], Deimling [3], Chandrasekher [2] and the references therein. Recently fixed point principles in Banach algebras due to Dhage [5, 6,]. The motivation of this paper due work of EL-Sayed A.M.A and Hashem H.H.G [4]. In this paper we modify positive solution of that problem [4] by using Dhage itration method. We prove the existence, approximations and positive solutions of a certain quadratic integral equation via an algorithm based on successive ap-proximations under partially Lipschitz and compactness conditions by using Dhage iteration method.

Given a closed and bounded interval J = [0; T], of the real line R, T > 0

$$x(t) = a(t) + g(t, x(t)) \int_0^t k(t,s) f(s, x(s)) ds \quad (1)$$

Where $f, g: J \times \mathbb{R} \to \mathbb{R}$, $k: J \times J \to \mathbb{R}$, $a: J \to \mathbb{R}$ are continuous functions.

By a solution of the NQIE (1) we mean a function $x \in C(J; R)$ that satisfies the equation (1) on J, where C(J ; R) is the space of continuous real-valued functions defined on J.

The NQIE (1) is well-known in the literature and here we find existence as well as approximations of positive solution of NQIE (1) by using work of Dhage [5]. If $k(t,s) f(s, x(s)) ds = 1$ for all $t \in J$ and $x \in R$ the NQIE (1) reduces to nonlinear functional equation.

$$x(t) = g(t, x(t)), \ t \in J \quad (2)$$

and if $a(t) = 0$ and $g(t, x(t)) = 1$ for all $t \in J$ and $x \in R$, it is reduced to nonlinear usual Volterra integral equation.

$$x(t) = \int_0^t k(t,s) f(s, x(s)) ds \quad (3)$$

Therefore, the NQIE (1)is general and the results of this paper include the existence, approximations results and positive solutions.

Auxiliary Results

Let E denote a partially ordered real normed linear space with an order relation $\leq$ and the norm $\|.\|$ .It is known that E is regular if $\{x_n\}_{n \in N}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$ then $x_n \leq x^*$ resp. $x_n \geq x^*$ for all $n \in$ N.

We need the following definitions.

Definition 2.1

A mapping $T: E \to E$ is called isotone or nondecreasing if it preserves the order relation $\leq$, that is if $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in E$.

Definition 2.2

A mapping $T: E \to E$ is called partially continuous at a Point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|T_x - T_a\| < \epsilon$ x is comparable to a and $\|x - a\| < \delta$. T called partially continuous on E if it is partially continuous at every point of it. It is clear that if T is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3

A mapping $T: E \to E$ is called partially bounded if T (C) is bounded for every chain C in E. T is called uniformly partially bounded if all chains T (C) in E are bounded by a unique constant. T is called bounded if T (E) is a bounded subset of E.

Definition 2.4

A mapping $T : E \to E$ is called partially compact if T(C) is a relatively compact subset of E for all totally ordered sets or chains C in E. T is called uniformly partially compact if T (C) is a uniformly partially bounded and partially compact on E. T is called partially totally bounded if for any totally ordered and bounded subset C of E, T (C) is a relatively compact subset of E. If T is partially continuous and partially totally bounded, then it is called partially completely continuous on E.

Definition 2.5

The order relation $\leq$ and the metric d on a non-empty set E are said to be compatible if $\{x_n\}_{n\in N}$ is a monotone that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in N}$ of $\{x_n\}_{n\in N}$ converges to x^* implies that the whole sequence $\{x_n\}_{n\in N}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \leq \|.\|)$, the order relation $\leq$ and the norm $\|.\|$ are said to be compatible if $\leq$ and the metric d defined through the norm $\|.\|$ are compatible. Clearly, the set R of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|.|$ has this property.Similarly, the finite dimensional Euclidean space R^n with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.6

A upper semi-continuous and nondecreasing function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ is called a D-function provided $\psi(0) = 0$.Let $(E, \leq, \|.\|)$ be a partially ordered normed linear space. A mapping $T : E \to E$ is called partially nonlinear D-Lipschitz if there exists a D-function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|T_x - T_y\| \leq \psi(\|x - y\|) \quad (4)$$

for all comparable elements $x; y \in E$.If $\psi(r) = kr;\ k > 0$, then T is called a partially Lipschitz with a Lipschitz constant k .Let $(E, \leq, \|.\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E | x \geq \theta \text{ where } \theta \text{ is zero element of } E\}$$

and

$$K = \{E^+ \subseteq E | uv \in E^+ \text{ for all } uv \in E^+\} \quad (5)$$

The elements of the set K are called the positive vectors in E.

Lemma 2.7

If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \leq v_1$ and $u_2 \leq v_2$ Then $u_1u_2 \leq v_1v_2$

Definition 2.8

An operator T: E →E is said to be positive if the range R T of T is such that R T ⊆ K. The Dhage iteration method is embodied in the following hybrid fixed point theorem which is a useful tool in this paper.

Theorem 2.9

Let $(E, \leq, \|.\|)$ be a regular partially ordered complete normed linear algebra such that the order relation $\leq$ and the norm $\|.\|$ in E are com-patible in every compact chain of E. Let $A; B : E \to K$ be two nondecreasing operators such that

1. A is partially bounded and partially nonlinear D-Lipschitz with D-function ψ_A
2. B is partially continuous and uniformly partially compact, and
3. $M\,\psi_A(r) < r;\ r > 0$, where and $M = \{\sup\|B(C)\|\ K: C \text{ is a chain in } E\}$ and
4. there exists an element $x_0 \in X$ such that $x_0 \leq Ax_0Bx_0$ or $x_0 \geq Ax_0Bx_0$. be a regular partially ordered complete normed line Then the operator equation

$$AxBx = x \quad (6)$$

has a positive solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = Ax_nBx_n$, $n = 0; 1 \ldots$. converges monotonically to x^*

Main Results

The NQIE (1) is considered in the function space $C(J; R)$ of continuous real-Valued functions defined on J. We define a norm $\|.\|$ and the order relation $\leq$ in $C(J; R)$ by

$$\|x\| = \sup_{t\in J}|x(t)| \quad (7)$$

And

$$x \leq y \Leftrightarrow x(t) \leq y(t) \quad (8)$$

for all $t \in J$ respectively. Clearly, $C(J; R)$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above par-tially order relation $\leq$ It is known that the partially ordered Banach algebra $C(J; R)$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzella-Ascolli theorem

Lemma 3.1

Let $(C(J; R, \leq, \|.\|)$ be a partially ordered Banach space with The norm $\|.\|$ and the order relation $\leq$ defined by (7) and (8) respectively. Then $\|.\|$ and $\leq$ are compatible in every partially compact subset of C (J;R).

We need the following definition in what follows

Definition 3.2

A function u ∈ C(J;R) is said to be a lower solution of the NQIE (1) if it satisfies

$$u(t) = a(t) + g(t, u(t)) \int_0^t k(t,s) f(s, u(s)) ds$$

for all t ≤ J Similarly, a function v ∈ C(J;R) is said to be a lower solution of the NQIE (1) if it satisfies the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

(A_0) a defines a continuous function $a : J \to \mathbb{R}_+$

(A_1) g defines a function $g: J \times \mathbb{R} \to \mathbb{R}$.

(A_2) There exists a real number $M_g > 0$ such that $g(t; x) \leq M_g$ for all t ∈ J and $x \in R$.

(A_3) There exists a D-function ϕ such that $0 \leq g(t; x) - g(t; y) \leq \phi(x - y)$ for all $t \in J$ and $x; y \in R, x \geq y$

(B_1) f defines a function $f : J \times \mathbb{R} \to \mathbb{R}$

(B_2) There exists a real number $M_f > 0$ such that $f(t; x) \leq M_f$ for all $t \in J$ and $x \in R$.

(B_3) f(t; x) is nondecreasing in x for all t ∈ J.

(B_4) The NQIE (1) has a lower solution $u \in C(J; R)$

Theorem 3.3

Assume that hypotheses (A0)-(A3) and (B1)-(B_4) hold. Furthermore, assume that,

$$(M_f KT)\psi_g(r) + \|a\| < r, r > 0 \quad (9)$$

then the NQIE has a positive solution x^* defined on J and the sequence $\{x_n\}_{n\in N}$ of successive approximations defined by.

$$x_{n+1}(t) = a(t) + g(t, x_n(t)) \int_{t_0}^t k(t,s) f(s, x_n(s)) ds, t \in J \quad (10)$$

where, $x_0 = u$ converges monotonically to x^*

Proof

Set $E = C(J;R)$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|.\|$ and the order relation $\leq$ in E. Define two operators A and B on E by

$$Ax(t) = g(t;\ x(t));\ t \in J \quad (11)$$

$$Bx(t) = \int_{t_0}^{t} k(t,s)f(s, x_n(s))ds \quad (12)$$

From the continuity of the integral and the hypotheses (A0)-(A1) and (B1), it follows that A and B define the maps $A; B : E \to K$. Now by definitions of the operators A and B, the NQIE (1) is equivalent to the operator equation

$$Ax(t).Bx(t) = x(t);\ t \in J \quad (13)$$

We shall show that the operators A and B satisfy all the conditions of Theorem 2.9. This is achieved in the series of following steps.

Step I

A and B are nondecreasing on E. Let $x;\ y \in E$be such that $x \geq\ y$. Then by hypothesis (A2), we obtain

$$\begin{aligned} Ax(t) &= a(t) + g(t;\ x(t) \\ &\geq a(t) + g(t;\ y(t)) \\ &= Ay(t) \end{aligned}$$

for all $t \in J$. This shows that A is nondecreasing operator on E into E. Similarly using hypothesis (B3), it is shown that the operator B is also nonde-creasing on E into itself. Thus, A and B are nondecreasing positive operators on E into itself.

Step II

A is partially bounded and partially D-Lipschitz on E.

Let $x \in E$ be arbitrary. Then by (A2)

$$\begin{aligned} |Ax(t)| &\leq |a(t) + g(t;\ x(t))\ | \\ &\leq |a(t)| + |g(t;\ x(t))| \\ &\leq \|a\| + M_g \end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|Ax\| \leq \|a\| + Mg$ and so, A is bounded. This further implies that A is partially bounded on E. Next, let $x;\ y \in E$ be such that $x \geq y$. Then, by hypothesis (A3),

$$\begin{aligned} |Ax(t) - Ay(t)| &= |a(t) + g(t;\ x(t)) - a(t) - g(t;\ y(t))\ | \\ &\leq |\ g(t;\ x(t)) - g(t;\ y(t))\ | \\ &\leq \phi(|x(t) - y(t)|) \\ &\leq \phi(x - y) \end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|Ax - Ay\| \leq \phi\ (\|x - y\|)$ for all $x;\ y \in E$ with $x \geq y$. Hence A is a partially nonlinear D-Lipschitz on E which further implies that A is a partially continuous on E.

Step III

B is partially continuous on E Let $\{x_n\}_{n\in N}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in N$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{x\to\infty} Bx(t) &= \lim_{x\to\infty} \int_0^t k(t,s)f(s,x(s))ds \\ &= \int_{t_0}^t k(t,s)\left[\lim_{x\to\infty} f(s,x(s))\right] ds \\ &= \int_{t_0}^t k(t,s)f(s,x(s))ds \\ &= Bx(t) \end{aligned}$$

for all $t \in J$. This shows that Bx_n converges monotonically to Bx pointwise on J. Next, we will show that $\{Bx_n\}_{n\in N}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$.

$$\begin{aligned} |Bx_n(t_2) - Bx_n(t_1)| &\leq \left|\int_0^{t_2} k(t_1,s)f(s,x(s))ds - \int_0^{t_2} k(t_2,s)f(s,x(s))ds\right| \\ &\leq \left|\int_0^T |k(t_1,s) - k(t_2,s)|\ M_f ds\right| \\ &\leq KM_f|t_1 - t_2| \\ &\to 0\ as\ t_1 - t_2 \to 0 \end{aligned}$$

uniformly for all $n \in N$. This shows that the convergence $Bx_n \to Bx$ is uniform and hence B is partially continuous on E.

Step IV

B is uniformly partially compact operator on E. Let C be an arbitrary chain in E. We show that B(C) is a uniformly bounded and equicontinuous set in E. First we show that B(C) is uniformly bounded. Let $y \in B(C)$ be any element. Then there is an element $x \in C$ be such that $y = Bx$. Now, by hypothesis (B2),

$$\begin{aligned} y(t) &\leq \int_{t_0}^t k(t,s)\left|f(s,x(s))\right|ds \\ &\leq KM_fT \\ &\leq r \end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|y\| = \|Bx\| \leq r$ for all $y \in B(C)$. Hence, B(C) is a uniformly bounded subset of E. Moreover, $\|B(C)\| \leq r$ for all chains C in E.Hence, B is a uniformly partially bounded operator on E.

Next, we will show that B(C) is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$ Then, for any $y \in B(C)$, one has,

$$\begin{aligned} |y(t_2) - y(t_1)| &= |Bx(t_2) - Bx(t_1)| \\ &\leq \left|\int_0^t k(t_1,s)f(s,x(s))ds - \int_0^{t_1} k(t_2,s)f(s,x(s))ds\right| \\ &\leq |k(t_1,s) - k(t_2,s)| \int_{t_1}^{t_2}|f(s,x(s))ds| \leq KM_f|t_1 - t_2| \\ &\to 0\ as\ t_1 - t_2 \to 0 \end{aligned}$$

uniformly for all $y \in B(C)$. Hence B(C) is an equicontinuous subset of E. Now, B(C) is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, B is a uniformly partially compact operator on E into itself.

Step V

u satisfies the operator inequality $u \leq AuBu$. By hypothesis (B4), the NQIE (1) has a lower solution u defined on J. Then,

We have

$$u(t) = a(t) + g(t,u(t)) \int_0^t k(t,s)f(s,u(s))ds \quad (14)$$

for all $t \in J$ From definitions of the operators A and B it follows that

$u(t) \leq Au(t)Bu(t)$ for all $t \in J$.

Hence $u \leq AuBu$.

Step VI

D-function ϕ satisfies the growth condition $M\phi(r) < r;\ r > 0$. Finally, the D-function Φ of the operator A satisfies the inequality given in hypothesis (d) of Theorem 2.9, viz.

$$M\psi_A(r) \leq (KM_fT + \|a\|)\phi(r) < r, \text{ for all } r > 0$$

Thus A and B satisfy all the conditions of Theorem 2.9 and we apply it to Conclude that the operator equation $Ax\,Bx = x$ has a solution. Consequently the integral equation and the NQIE (1) has a solution x^* defined on J.Fur-theremore, $\{x_n\}_{n\in N}$ the sequence of successive approximations defined by (10) converges monotonically to x^*.This completes the proof.

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